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The Method of Partial Fractions
Math 121 Calculus II
Spring 2015

Rational functions. Recall that a rational function is a quotient of two polynomials such as

$$\frac{f(x)}{g(x)} = \frac{3x^5 + x^3 + 16x^2 - 42x - 60}{x^3 + x^2 - 4x - 4}.$$

The method of partial fractions can be used to integrate rational functions. It's a fairly complicated method and applying it takes time, but it works!

We know how to integrate a few simple rational functions. This method will take a complicated rational function like the example above and express it as a sum of the simple rational functions we can already integrate, like these:

$$\int \frac{dx}{x} = \ln|x| + C, \quad \int \frac{dx}{x^n} = \frac{-1}{(n-1)x^{n-1}} + C \quad \text{for } n > 1, \quad \text{and} \int \frac{dx}{1+x^2} = \arctan x + C$$

These can be generalized to these three more useful identities

$$\begin{aligned} \int \frac{dx}{ax+b} &= \frac{1}{a}\ln|ax+b| + C \\ \int \frac{dx}{(ax+b)^n} &= \frac{-1}{a(n-1)(ax+b)^{n-1}} + C \quad \text{for } n > 1 \\ \int \frac{dx}{a^2x^2+b^2} &= \frac{1}{ab}\arctan\frac{ax}{b} + C \end{aligned}$$

Using the method called completing the square, that last equation can be further generalized to integrate the reciprocal of any irreducible quadratic function.

Results from algebra. This method depends on some algebraic facts about polynomials, namely, the Fundamental Theorem of Algebra and the decomposition of rational functions into partial fractions. The term "partial fractions" just means simpler rational functions.

The FTA. The Fundamental Theorem of Algebra (FTA) concerns factoring a polynomial into linear factors. An example of such a factoring is $x^2 - 8x + 15 = (x-5)(x-3)$. It can also be stated in terms of roots of the polynomial. For the example, the quadratic polynomial $x^2 - 8x + 15$ has two roots, $x = 5, 3$.

The FTA states that over the complex numbers, every n^{th} degree polynomial can be factored into n linear factors. In terms of roots, it says that every n^{th} degree polynomial has exactly n roots, but the roots may be complex numbers, and multiplicities have to be counted.

For an example of counting multiplicities, the cubic polynomial $x^3 + x^2 - x - 1$ factors into linear factors as $(x+1)^2(x-1)$. The factor $x+1$ appears twice. Its roots are $+1$ and -1, but -1 has multiplicity 2, i.e, it's a double root.

For an example of complex roots, the polynomial x^2+1 factors as $(x+i)(x-i)$, and has two roots $\pm i$. Here, i is $\sqrt{-1}$, the imaginary unit.

We don't have to get involved with complex numbers. Instead, we'll take a form of the FTA that only mentions real numbers.

In that form, the FTA states that every n^{th} degree polynomial can be factored over the real numbers into linear factors and irreducible quadratic factors. An irreducible quadratic polynomial is one like x^2+1 that has no real roots. You can tell if a quadratic polynomial ax^2+bx+c is irreducible by looking at its *discriminant* b^2-4ac. If the discriminant is positive, then the polynomial has two real roots and it factors; if the discriminant is 0, then the polynomial has a double real root and it factors as the square a linear polynomial; but if the discriminant is negative, then it has two complex roots and it doesn't factor, that is, it's irreducible.

The FTA is used for partial fractions in order to factor the denominator of the rational function.

Decomposition of rational functions into partial fractions. There are a few steps to accomplish this decomposition. First divide the denominator into the numerator, then factor the denominator, next write the rational function as a sum of partial fractions with undetermined constants, and finally determine those constants.

Example 1. *Step 1.* The first step in the decomposition is to reduce the problem to the case where the numerator has a lower degree than the denominator. Let's take our example $\dfrac{f(x)}{g(x)}=\dfrac{3x^5+x^3+16x^2-42x-60}{x^3+x^2-4x-4}$. You can use long division to divide $g(x)$ into $f(x)$. In this case, divide x^3+x^2-4x-4 into $3x^5+x^3-42x-60$. You'll get $3x^2-3x+16$ with a remainder of $10x+4$. Therefore,

$$\frac{3x^5+x^3+16x^2-42x-60}{x^3+x^2-4x-4}=3x^2-3x+16+\frac{10x+4}{x^3+x^2-4x-4}.$$

Step 2. Next factor the denominator. By the FTA we know there is a factorization. Finding the factorization for high degree polynomials is difficult, but in this example, we can do it.

$$x^3+x^2-4x-4=(x-4)(x-1)(x+1)$$

Step 3. Now comes the theory for the decomposition. It says that any rational function whose numerator has a lower degree than the denominator can be written as a sum of simpler rational functions, the "partial fractions", where each partial fraction has as denominator a factor of the original denominator or a power of that factor if that factor appears with multiplicity greater than 1. The numerators of the partial fractions have lower degrees than the denominators. For our example, this says that

$$\frac{10x+4}{x^3+x^2-4x-4}=\frac{A}{x-4}+\frac{B}{x-1}+\frac{C}{x+1}$$

where, since the denominators have degree 1, the numerators have to have degree 0, that is, A, B, and C are constants.

Step 4. Determine the numerators. Clear the denominators from the equation by multiplying both sides by $x^3+x^2-4x-4=(x-4)(x-1)(x+1)$. You'll get

$$10x+4=A(x-1)(x+1)+B(x-4)(x+1)+C(x-4)(x-1).$$

There are various ways to continue from here to determine A, B, and C. Since the equation is true for all values of x, one way to continue is to set x to convenient values, and the most convenient values are the roots of the polynomial: 4, 1, and -1.

If you set $x = 4$, then the equation says $44 = 15A$, so $A = \frac{44}{15}$.

If you set $x = 1$, then the equation says $14 = -6B$, so $B = -\frac{7}{3}$.

And if you set $x = -1$, then the equation says $-6 = 10C$, so $C = -\frac{3}{5}$.

Therefore,

$$\frac{10x+4}{x^3+x^2-4x-4} = \frac{44/15}{x-4} + \frac{-7/3}{x-1} + \frac{-3/5}{x+1}.$$

Complete the integration. Now that the complicated rational function has been written as a sum of simpler partial fractions, we can integrate it.

$$\begin{aligned}
& \int \frac{3x^5+x^3+16x^2-42x-60}{x^3+x^2-4x-4}\,dx \\
= & \int \left(3x^2-3x+16+\frac{10x+4}{x^3+x^2-4x-4}\right) dx \\
= & \int \left(3x^2-3x+16+\frac{44/15}{x-4}+\frac{-7/3}{x-1}+\frac{-3/5}{x+1}\right) dx \\
= & \; x^3-\frac{3}{2}x^2+16x+\frac{44}{15}\ln|x-4|-\frac{7}{3}\ln|x-1|-\frac{3}{5}\ln|x+1|+C
\end{aligned}$$

This example was chosen to show the method, but there can be complications. One complication occurs when there is a multiple root. A more difficult complication occurs when one of the factors is an irreducible quadratic polynomial. That eventually leads to an answer involving arctangents.

We'll look at a couple more examples to see some of these complications.

Example 2. $\displaystyle\int \frac{2x^4\,dx}{x^3+x^2-x-1}$. First divide the denominator into to the numerator to see that

$$\frac{2x^4}{x^3+x^2-x-1} = 2x-2+\frac{4x^2+1}{x^3+x^2-x-1}.$$

Next, the denominator factors as $x^3+x^2-x-1 = (x+1)^2(x-1)$. The repeated factor of $(x+1)^2$ will give us two terms in the partial fraction decomposition as follows.

$$\frac{4x^2+1}{x^3+x^2-x-1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

Determine the constants A, B, and C. First clear the denominators.

$$4x^2+1 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

The two roots ± 1 give us two convenient values of x to determine two of these constants.

Setting $x = 1$, we get $5 = 4C$, so $C = \frac{5}{4}$.

Setting $x = -1$, we get $5 = -2B$, so $B = -\frac{5}{2}$.

There are no more roots, but we can set x to any other value, say $x = 0$, and we find $1 = -A - B + C$, and since we know B and C, therefore $A = \frac{11}{4}$. Thus,

$$\frac{2x^4\,dx}{x^3 + x^2 - x - 1} = 2x - 2 + \frac{11/4}{x+1} + \frac{-5/2}{(x+1)^2} + \frac{5/4}{x-1}.$$

We can now integrate the original rational function.

$$\begin{aligned} \int \frac{2x^4\,dx}{x^3 + x^2 - x - 1} &= \int \left(2x - 2 + \frac{11/4}{x+1} + \frac{-5/2}{(x+1)^2} + \frac{5/4}{x-1}\right) dx \\ &= x^2 - 2x + \frac{11}{4}\ln|x+1| + \frac{5/2}{2(x+1)} + \frac{5}{4}\ln|x-1| + C \end{aligned}$$

Example 3. $\displaystyle\int \frac{x^2 + 2x + 5}{(x^2+1)(x-3)}\,dx$. The numerator already has a smaller degree than the denominator, so we can skip that step, and the denominator is already factored as the product of the irreducible quadratic $x^2 + 1$ and the linear factor $x - 3$. The partial fraction decomposition has two parts, the first with denominator $x^2 + 1$, but its numerator won't be a constant but an undetermined linear polynomial $Ax + B$.

$$\frac{x^2 + 2x + 5}{(x^2+1)(x-3)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3}$$

Now to determine A, B, and C. We'll clear the denominators.

$$x^2 + 2x + 5 = (Ax + B)(x - 3) + C(x^2 + 1)$$

Although we could use the root $x = 3$ as a convenient value and determine what C is, we'd still have to find A and B somehow. Let's use another method from algebra. Rewrite the right hand side of the equation as

$$x^2 + 2x + 5 = (A + C)x^2 + (-3A + B)x + (-3B + C).$$

The only way two polynomials can be equal is if they have the same coefficients. That gives us three simultaneous linear equations in three unknowns to solve.

$$\begin{cases} 1 = A + C \\ 2 = -3A + B \\ 5 = -3B + C \end{cases}$$

You can solve these equations the unknowns A, B, and C to find that $A = 2$, $B = -1$, and $C = 2$. Now we can continue the integration.

$$\int \frac{x^2 + 2x + 5}{(x^2+1)(x-3)}\,dx = \int \frac{2x - 1}{x^2 + 1}\,dx + \int \frac{2}{x - 3}\,dx$$

The second integral is just $2\ln|x - 3|$. We can split the first integral into two parts. The first part is

$$\int \frac{2x}{x^2 + 1}\,dx = \ln|x^2 + 1|$$

which you can guess or find with the help of a substitution $u = x^2 + 1$, while the second part is

$$\int \frac{-1}{x^2+1}\,dx = -\arctan x.$$

Thus, our original integral evaluates as

$$\ln|x^2+1| - \arctan x + 2\ln|x-3| + C.$$

See the text for some examples that are more complicated than these.

Weierstrass's universal t-substitution. This is a substitution that converts and any function built out of trig functions and the four arithmetic operations into a rational function like the ones we just looked at.

Suppose we have an integral of such a function, say $\int \frac{2+\sin\theta}{3+\cos\theta}\,d\theta$. Weierstrass's t substitution makes $t = \tan\theta/2$. The full set of formulas for this substitution is

$$\begin{aligned} t &= \tan\frac{\theta}{2} & \sin\theta &= \frac{2t}{1+t^2} \\ \theta &= 2\arctan t & \cos\theta &= \frac{1-t^2}{1+t^2} \\ d\theta &= \frac{2}{1+t^2}\,dt & \tan\theta &= \frac{2t}{1-t^2} \end{aligned}$$

Our example integral then becomes

$$\int \frac{2+\dfrac{2t}{1+t^2}}{3+\dfrac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{2(t^2+t+1)}{(t^2+2)(t^2+1)}\,dt$$

The methods described above finish off this integral.

$$= \int \frac{1-t}{t^2+2}\,dt + \int \frac{t}{t^2+1}\,dt = \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} - \tfrac{1}{2}\ln(t^2+2) + \tfrac{1}{2}\ln(t^2+1) + C$$

where $t = \tan\theta/2$.

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