Power Series
Math 121 Calculus II
Spring 2015

Introduction to power series. One of the main purposes of our study of series is to understand power series. A power series is like a polynomial of infinite degree. For example,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

is a power series. We'll look at this one in a moment.

Power series have a lot of properties that polynomials have, and that makes them easy to work with. Also, they're general enough to represent lots of important functions like e^x, $\ln x$, $\sin x$, and $\cos x$.

Let's look at $1 + x + x^2 + \cdots + x^n + \cdots$. For a fixed value of x, this is just a geometric series, and we know that it sums to $\dfrac{1}{1-x}$ when $x \in (-1, 1)$. For values of x outside the interval $(-1, 1)$, the series diverges. Thus,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } x \in (-1, 1).$$

The interval where a power series converges is called the *interval of convergence*.

Since we know the series for $\dfrac{1}{1-x}$, we can use it to find power series for some other functions. For instance, if we substitute $-x$ for x we get

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots \qquad \text{for } x \in (-1, 1).$$

If, instead, we substitute $2x$ for x we get

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots \qquad \text{for } x \in (-\tfrac{1}{2}, \tfrac{1}{2}),$$

and $-x^2$ for x gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots \qquad \text{for } x \in (-1, 1).$$

Two of the important properties of polynomials that are shared with power series is that they can be differentiated and integrated term by term. That is, if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

for x in some interval, then, differentiating,

$$f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots$$

for x in the same interval; also, integrating,

$$\int f(x)\,dx = a_0 x + a_1\,\frac{x^2}{2} + \cdots + a_n\,\frac{x^{n+1}}{n+1} + \cdots + C$$

again for x in that interval. Integration works not only for indefinite integrals (as just written), but also for definite integrals.

Thus, integrating the series for $\dfrac{1}{1+x}$ between 0 and x, we get

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{for } x \in (-1,1).$$

We get the power series for $\arctan x$ by integrating the series for $\dfrac{1}{1+x^2}$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{6} + \cdots \qquad \text{for } x \in (-1,1).$$

We'll need more theory to develop the following power series for the three important functions e^x, $\cos x$, and $\sin x$. We'll do that next time. It leads to the important results that for all x,

$$\begin{array}{rcccccccccccccc}
e^x & = & 1 & + & x & + & \frac{x^2}{2!} & + & \frac{x^3}{3!} & + & \frac{x^4}{4!} & + & \frac{x^5}{5!} & + & \cdots \\
\cos x & = & 1 & & & - & \frac{x^2}{2!} & & & + & \frac{x^4}{4!} & & & - & \cdots \\
\sin x & = & & & x & & & - & \frac{x^3}{3!} & & & + & \frac{x^5}{5!} & & \cdots .
\end{array}$$

Note that the terms of the $\cos x$ series are the even terms of the e^x series, but the $\cos x$ series alternates sign, and the terms of the $\sin x$ series are the odd terms of the e^x series, and, again, the $\sin x$ series alternates sign.

Power series centered at numbers other than 0. The power series mentioned above can be translated to other numbers if we make a linear substitution.

For example, take the power series for $\ln(1+x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{for } x \in (-1,1)$$

and substitute $u = 1 + x$, so $x = u - 1$. Then

$$\ln u = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(u-1)^n}{n} = (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \cdots + (-1)^{n+1}\frac{(u-1)^n}{n} + \cdots \qquad \text{for } u \in (0,2).$$

This new series expresses $\ln u$ as a series of powers of $u-1$. This power series is said to be centered at 1. Note that the interval of convergence is translated by 1.

For another example, take our first power series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } x \in (-1,1)$$

and substitute $u = x - 1$, so $x = u + 1$. Then we find

$$\frac{-1}{u} = 1 + (u+1) + (u+1)^2 + \cdots + (u+1)^n + \cdots \qquad \text{for } u \in (-2, 0).$$

This series is centered at -1.

For the most part, we'll use power series centered at 0, but sometimes power series centered at other numbers are useful.

In summary, a power series centered at a number a is of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots .$$

We'll see next that power series centered at a number a have intervals of convergence centered around a, that is, the series will converge if x lies between $a - r$ and $a + r$ where r is half the length of the interval. This number r is called the *radius of convergence*.

Finding the radius of convergence. There's a version of the ratio test which will usually be able to tell us what the radius of convergence of a power series is. It doesn't work for all possible power series, but it does for all the important ones.

Treat x as a constant and apply the usual ratio test in conjunction with the absolute convergence test on a power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at a. Suppose that the absolute value of the ratio of the next term to the present term has a limit L.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x-a| = |x-a| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Let r be the reciprocal of $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then $L = |x-a|/r$.

Now, if $L < 1$, the series absolutely converges, but if $L > 1$ it diverges. In terms of r, that says $|x-a| < r$, then the series absolutely converges, but if $|x-a| > r$ it diverges by the term test. In other words the power series converges in an interval centered at a with radius r. Thus, we've proven the following theorem.

Theorem 1 (Abel)**.** If the ratio $|a_{n+1}/a_n|$ of the absolute values of the coefficients of a power series has a limit, then the reciprocal of that limit is the radius of convergence of the power series. More precisely, if the series is $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$, and $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{r}$, then the series absolutely converges for $x \in (a-r, a+r)$, and the series diverges for $|x-a| > r$.

Note that this theorem says nothing about convergence at the endpoints of the interval of convergence. Those two values, $x = a \pm r$, need to be checked separately for convergence.

Example 2. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-3)^n$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{(n+1)2^{n+1}} \Big/ \frac{1}{n2^n} = \frac{n}{2(n+1)} \to \frac{1}{2}$$

So the radius of convergence is 2, and the interval of convergence goes from 1 to 5.

Now let's check the two endpoints for convergence. At $x = 5$ the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which we know diverges. At $x = 2$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which we know converges. Thus, the interval of convergence for this series is the half open interval $[2, 5)$.

Extension of the theory to complex numbers. Everything mentioned so far applies only to real numbers, but it can all be extended to complex numbers. In particular, consider the last theorem. Let a complex power series be $\sum_{n=0}^{\infty} a_n (z-a)^n$, where a and each a_n is a complex number and z is a complex variable. If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{r}$, then the series converges when $|z - a| < r$, that is, it converges in a circle centered at the point a with radius r. (Note that the limit is the limit of real numbers and r is a real number.)

For example, the series in the last example converges inside the circle of radius 2 centered at 3. It will diverge outside that circle. It will converge for some points on the circle and diverge for others.

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