

Rules and methods for integration
 Math 121 Calculus II
 Spring 2015

We've covered the most important rules and methods for integration already. We'll look at a few special-purpose methods later on.

The fundamental theorem of calculus. This is the most important theorem for integration. It tells you that in order to evaluate an integral, look for an antiderivative. If F is an antiderivative of f , meaning that f is the derivative of F , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Because of this FTC, we write antiderivatives as indefinite integrals, that is, as integrals without specific limits of integration, and when F is an antiderivative of f , we write

$$\int f(x) dx = F(x) + C$$

to emphasize that there are lots of other antiderivatives that differ from F by a constant.

Linearity of integration. This says that integral of a linear combination of functions is that linear combination of the integrals of those functions. In particular, the integral of a sum (or difference) of functions is the sum (or difference) of the integrals of the functions,

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

and the integral of a constant times a function is that constant times the integral of the function

$$\int cf(x) dx = c \int f(x) dx.$$

The power rule. For $n \neq -1$, $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$,

but for $n = -1$, $\int x^{-1} dx = \ln|x| + C$.

With linearity and the power rule we can easily find the integral of any polynomial.

The exponential function. $\int e^x dx = e^x + C$. Later we'll look at trigonometric functions and see what their derivatives and integrals are.

The method of substitution. This method comes from the chain rule for derivatives. It is not a rule so much as a method that often helps in evaluating integrals. If $u = f(x)$ so that $du = f'(x) dx$, then

$$\int g(f(x))f'(x) dx = \int g(u) du.$$

Remember that when you use this method all the instances of the original variable x must be accounted for and replaced by expressions involving u . You can even change the limits of integration so that you don't have to return to the original variable before the final evaluation. Here are a couple of suggestions when to use substitution.

1). If you see a composition somewhere in the integral, then it may help to substitute for the inner function. For instance, if $e^{\sqrt{x}}$ appears in the integral, maybe the substitution $u = \sqrt{x}$ will help. In order for the substitution to actually work, however, the derivative $\frac{du}{dx}$ has to appear as a factor of the integrand.

2). If you have a quotient for an integrand, and the derivative of the denominator is the numerator or a factor of the numerator, then, by all means, use the substitution where you let u be the denominator.

The method of integration by parts. This method follows from the product rule for derivatives. It says

$$\int u dv = uv - \int v du.$$

If the integrand is a product, then maybe integration by parts will help. When you select u and dv , try to choose u so that its derivative is simpler than it is, and dv so that you can integrate it to find what v is.

Even when the integrand isn't a product, integration by parts might help if the derivative of the integrand is simpler. Try letting u be the whole integrand, and take dv to be dx .

When one application of integration by parts only simplifies the integral a bit, you may have to use parts again (and again).

Other methods. There are a couple other methods we'll discuss later, but on the whole they aren't as important as the ones we've seen already. One involves some special substitutions called trig subs. Another is the method of partial fractions; it's used when the integrand is a rational function, that is, a quotient of two polynomials.

For many integrals, a combination of methods is needed. Perhaps an initial substitution will simplify the integral a bit, then one or more methods will complete the evaluation.

In many cases, tables of integrals will save you some time in evaluating integrals, and they will reduce the likelihood of minor errors. There are symbolic integrators built in to many calculators, and several are on the web as well.

Even with all this help, some functions, although integrable (because they're continuous) cannot be found as elementary functions (expressed in terms of polynomials, trig functions, inverse trig functions, logs, and exponentiation). Even a simple-looking integral such as $\int \sqrt{1+x^3} dx$ is not an elementary function. For such integrals numerical methods are required. For definite integrals, they approximate the value of the integral to as much accuracy as you require. For indefinite integrals, they construct tables of values for an antiderivative, again to as much accuracy as you require.

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