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Rotations and complex eigenvalues

Math 130 Linear Algebra

D Joyce, Fall 2015

Rotations are important linear operators, but they don't have real eigenvalues. They will, however, have complex eigenvalues.

Eigenvalues for linear operators are so important that we'll extend our scalars from $\mathbf{R}$ to $\mathbf{C}$ to ensure there are enough eigenvalues.

Two nice things about the field C of complex numbers. The Fundamental Theorem of Algebra states that if a polynomial with coefficients in $\mathbf{C}$ has degree n, then it has all n roots (when multiplicities are counted). Fields like $\mathbf{C}$ with that property are called *algebraically closed* fields.

In the early 1700s mathematicians noticed a connection between logarithms and the arctangent function. Euler explained the connection more simply using the complex exponential function by a formula now known as Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler used the power series to define the complex exponential function, and his formula directly directly follows by examining that power series and the series for cosine and sine. We'll occasionally use his formula.

Example 1. We'll look at general rotations in the next example , but let's warm up with a counterclockwise rotation by 90°. That's the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

Its characteristic polynomial is $\det(A - \lambda I)$ which equals

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

There are no real roots of this polynomial $\lambda^2 + 1$, only the imaginary roots $\pm i$. Thus this rotation has no real eigenvalues and no real eigenvectors.

How can we continue on? We can treat the matrix as a matrix over the complex numbers $\mathbf{C}$ instead of just the real numbers $\mathbf{R}$. Now it describes a linear transformation $\mathbf{C}^2 \to \mathbf{C}^2$. It has two complex eigenvalues, $\pm i$, that is, the spectrum for a 90° counterclockwise rotation is the set $\{i, -i\}$.

Let's find the eigenvalues for the eigenvalue $\lambda_1 = i$. We'll row-reduce the matrix $A - \lambda_1 I$.

$$A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

Thus, the solutions to this system, that is, the λ_1-eigenspace, is the set of vectors in $\mathbf{C}^2$ of the form $(z, w) = (iw, w)$ where w is an arbitrary complex number.

Likewise, you can show that the λ_2-eigenspace, where $\lambda_2 = -i$, consists of vectors $(z, w) = (-iw, w)$ where w is arbitrary.

Example 2. Eigenvalues of a general rotation in $\mathbf{R}^2$.

Recall that the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

describes a rotation of the plane by an angle of θ.

Let's find the eigenvalues of this generic rotation of the plane. The characteristic polynomial is $\det(A - \lambda I)$ which equals

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta.$$

We'll set that to 0 and solve for λ. We quickly run into problems, as

$$(\cos\theta - \lambda)^2 = -\sin^2\theta$$

has no real solutions. Thus, there are no real eigenvalues for rotations (except when θ is a multiple of π, that is the rotation is a half turn or the identity).

To get the missing eigenvalues, we'll treat the matrix as a matrix over the complex numbers $\mathbf{C}$ instead of just the real numbers $\mathbf{R}$. Then it describes a linear transformation $\mathbf{C}^2 \to \mathbf{C}^2$, and we can continue on.

$$\cos\theta - \lambda = \pm i \sin\theta$$

$$\lambda = \cos\theta \pm i \sin\theta = e^{\pm i\theta}$$

We get two complex eigenvalues. Each of these will have an associated eigenspace. Let's find the eigenspace for $\lambda_1 = \cos\theta + i\sin\theta$. We'll solve the equation $(A - \lambda_1)\mathbf{x} = \mathbf{0}$ by row-reducing the matrix $A - \lambda_1 I$.

$$\begin{aligned} A - \lambda_1 I &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - (\cos\theta + i\sin\theta)I \\ &= \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \\ &\sim \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Thus, the generic solution to this system is $(z, w) = (iw, w)$ where w is an arbitrary complex number.

Generally speaking, finding the complex eigenspaces for a rotation isn't as important as finding the eigenvalues.

Math 130 Home Page at
`http://math.clarku.edu/~ma130/`