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Norm and inner products in $\mathbf{C}^n$, and abstract inner product spaces

Math 130 Linear Algebra

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We've seen how norms and inner products work in $\mathbf{R}^n$. They can also be defined for $\mathbf{C}^n$. There's a wrinkle in the definition of complex inner products.

The norm of a complex vector v. We'll start with the norm for $\mathbf{C}$ which is the one-dimensional vector space $\mathbf{C}^1$, and extend it to higher dimensions.

Recall that if $z = x + iy$ is a complex number with real part x and imaginary part y, the complex conjugate of z is defined as $\overline{z} = x - iy$, and the absolute value, also called the norm, of z is defined as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\,\overline{z}}.$$

Now, if $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbf{C}^n$ where each v_i is a complex number, we'll define its norm $\|\mathbf{v}\|$ as

$$\begin{aligned} \|\mathbf{v}\| &= \|(v_1, v_2, \ldots, v_n)\| \\ &= \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2} = \sqrt{\sum_{k=1}^{n} |v_k|^2}. \end{aligned}$$

Note that if the coordinates of $\mathbf{v}$ all happen to be real numbers, then this definition agrees with the norm for real vector spaces.

Norms on $\mathbf{C}^n$ enjoy many of the same properties that norms on $\mathbf{R}^n$ do. For instance, the norm of any vector is nonnegative, and the only vector with norm 0 is the $\mathbf{0}$ vector. Also, norms are multiplicative in the sense that

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

when c is a complex number and $\mathbf{v}$ is a complex vector.

Furthermore, the triangle inequality for complex norms holds

$$\|\mathbf{v} - \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|.$$

We'll prove it later.

The inner product $\langle \mathbf{v}|\mathbf{w}\rangle$ of two complex vectors. We would like to have a complex inner product that (1) extends the real product, (2) is connected to the complex norm by the equation $\|\mathbf{v}\|^2 = \langle \mathbf{v}|\mathbf{v}\rangle$, and (3) has nice algebraic properties such as bilinearity.

In order to get property (2), we'll have to introduce a wrinkle into the definition. We cannot define $\langle (v_1, \ldots, v_n)|(w_1, \ldots, w_n)\rangle$ as $v_1w_1 + \cdots + v_nw_n$, because then $\langle (v_1, \ldots, v_n)|(v_1, \ldots, v_n)\rangle$ would equal $v_1^2 + \cdots + v_n^2$ which doesn't equal $|v_1|^2 + \cdots + |v_n|^2$. If we throw in a complex conjugate, however, it will work. That explains the following definition.

Definition 1. The standard complex inner product of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{C}^n$ is defined by

$$\begin{aligned} \langle \mathbf{v}|\mathbf{w}\rangle &= \langle (v_1, v_2, \ldots, v_n)|(w_1, w_2, \ldots, w_n)\rangle \\ &= v_1\overline{w_1} + v_2\overline{w_2} + \cdots + v_n\overline{w_n} = \sum_{k=1}^{n} v_k\overline{w_k} \end{aligned}$$

It follows that for each $\mathbf{v} \in \mathbf{C}^n$, our desired condition (2) above, holds

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}|\mathbf{v}\rangle.$$

Also, condition (1) holds. If $\mathbf{v}$ and $\mathbf{w}$ happen to be a real vectors, then their complex inner product is the same as their real inner product.

Most of the algebraic properties of complex inner products are the same as those of real inner product. For instance, inner products distribute over addition,

$$\langle \mathbf{u}|\mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}|\mathbf{v}\rangle + \langle \mathbf{u}|\mathbf{w}\rangle,$$

and over subtraction,

$$\langle \mathbf{u}|\mathbf{v} - \mathbf{w}\rangle = \langle \mathbf{u}|\mathbf{v}\rangle - \langle \mathbf{u}|\mathbf{w}\rangle,$$

and the inner product of any vector and the $\mathbf{0}$ vector is 0

$$\langle \mathbf{v} | \mathbf{0} \rangle = 0.$$

However, complex inner products are not commutative. Instead they have the property

$$\langle \mathbf{u} | \mathbf{v} \rangle = \overline{\langle \mathbf{v} | \mathbf{u} \rangle}.$$

Complex inner products are linear in their first argument. If c is a complex scalar, then

$$\langle c\mathbf{u} | \mathbf{v} \rangle = c \langle \mathbf{u} | \mathbf{v} \rangle$$

In for the second argument, we have instead

$$\langle \mathbf{u} | c\mathbf{v} \rangle = \overline{c} \langle \mathbf{u} | \mathbf{v} \rangle.$$

The complex conjugate of c comes from our definition where we use the complex conjugates of coordinates of the second vector.

In summary, complex inner products are not bilinear, but they are linear in the first argument and *conjugate linear* in the second argument.

Abstract linear spaces. So far, we've looked at the standard real inner product on $\mathbf{R}^n$ and the standard complex inner product on $\mathbf{C}^n$. Although we're primarily concerned with standard inner products, there are other inner products, and we should consider the generalization of these standard inner products. We'll call a vector space equipped with an inner product an inner product space.

We can make the definitions for abstract inner product spaces for both the real case and the complex case at the same time. In the definition, we'll take the scalar field F to be either $\mathbf{R}$ or $\mathbf{C}$.

Definition 2. An *inner product space* over F is a vector space V over F equipped with a function $V \times V \to F$ that assigns to vectors $\mathbf{v}$ and $\mathbf{w}$ in V a scalar denoted $\langle \mathbf{v} | \mathbf{w} \rangle$, called the *inner product* of $\mathbf{v}$ and $\mathbf{w}$, which satisfies the following four conditions:

(a). $\langle \mathbf{u} + \mathbf{v} | \mathbf{w} \rangle = \langle \mathbf{u} | \mathbf{w} \rangle + \langle \mathbf{u} | \mathbf{w} \rangle$,
(b). $\langle c\mathbf{v} | \mathbf{w} \rangle = c \langle \mathbf{v} | \mathbf{w} \rangle$,
(c). $\overline{\langle \mathbf{v} | \mathbf{w} \rangle} = \langle \mathbf{w} | \mathbf{v} \rangle$, and
(d). $\langle \mathbf{v} | \mathbf{v} \rangle > 0$ if $\mathbf{v} \neq \mathbf{0}$.

For an inner product space, the *norm* of a vector $\mathbf{v}$ is defined as $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v} | \mathbf{v} \rangle}$.

Note that when $F = \mathbf{R}$, condition (c) simply says that the inner product is commutative.

Properties (a) and (b) state that the inner product is linear in the first argument. Using those and (c), you can show that the inner product is conjugate linear in the second argument.

Condition (d) says that the norm $\|\mathbf{v}\|$ of a vector is always positive except in the one case the that $\mathbf{v} = \mathbf{0}$. From condition (b) you can infer $\|\mathbf{0}\| = 0$. Another property of norms is that $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$.

Finally, two more properties of inner products are the Cauchy-Schwarz inequality $|\langle \mathbf{v} | \mathbf{w} \rangle| \leq \|\mathbf{v}\| \, \|\mathbf{w}\|$, and the triangle inequality $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$. We'll prove them later.

Examples. The standard inner products on $\mathbf{R}^n$ and $\mathbf{C}^n$ are, of course, the primary examples of inner product spaces.

Our text describes some other inner product spaces besides the standard ones $\mathbf{R}^n$ and $\mathbf{C}^n$. One is a real inner product on the vector space of continuous real-valued functions on $[0, 1]$. Another is an inner product on $m \times n$ matrices over either $\mathbf{R}$ or $\mathbf{C}$. We'll discuss those briefly in class. There's another example of the vector space of complex-valued functions on the unit circle we won't have time for.

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