Subspaces of Vector Spaces
Math 130 Linear Algebra

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Subspaces. A *subspace* W of a vector space V is a subset of V which is a vector space with the same operations.

We've looked at lots of examples of vector spaces. Some of them were subspaces of some of the others. For instance, P_n, the vector space of polynomials of degree less than or equal to n, is a subspace of the vector space P_{n+1} of polynomials of degree less than or equal to $n+1$.

For a vector space to be a subspace of another vector space, it just has to be a subset of the other vector space, and the operations of vector addition and scalar multiplication have to be the same.

Perhaps the name "sub vector space" would be better, but the only kind of spaces we're talking about are vector spaces, so "subspace" will do.

Another characterization of subspace is the following theorem.

Theorem 1. A subset W of a vector space V is a subspace of V if and only if

(1) $\mathbf{0} \in W$;

(2) W is closed under vector addition, that is, whenever $\mathbf{w}_1$ and $\mathbf{w}_2$ belong to W, then so does $\mathbf{w}_1 + \mathbf{w}_2$ belong to W; and

(3) W is closed under scalar products, that is, whenever c is a real number and $\mathbf{w}$ belongs to W, then so does $c\mathbf{w}$ belong to W.

Yet another characterization of subspace is this theorem.

Theorem 2. A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under linear combinations, that is, whenever $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ all belong to W, then so does each linear combination $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_k$ of them belong to W.

This second characterization is equivalent to the first because, first, linear combinations are built from vector additions and scalar products, and, second, scalar products and vector additions are special cases of linear combinations.

Subspaces of the plane $\mathbf{R}^2$. Let's start by examining what it means to be a subspace of the vector space $\mathbf{R}^2$. This will be enough to see what the concept means.

First of all, there are a couple of obvious and uninteresting subspaces. One is the whole vector space $\mathbf{R}^2$, which is clearly a subspace of itself. A subspace is called a *proper* subspace if it's not the entire space, so $\mathbf{R}^2$ is the only subspace of $\mathbf{R}^2$ which is not a proper subspace.

The other obvious and uninteresting subspace is the smallest possible subspace of $\mathbf{R}^2$, namely the $\mathbf{0}$ vector by itself. Every vector space has to have $\mathbf{0}$, so at least that vector is needed. But that's enough. Since $\mathbf{0} + \mathbf{0} = \mathbf{0}$, it's closed under vector addition, and since $c\mathbf{0} = \mathbf{0}$, it's closed under scalar multiplication. This $\mathbf{0}$ subspace is called the *trivial* subspace since it only has one element.

So, ignoring those two obvious and uninteresting subspaces, we're left with finding all the rest, and they're the proper, nontrivial subspaces of $\mathbf{R}^2$.

And here they are. Take any line W that passes through the origin in $\mathbf{R}^2$. If you add two vectors in that line, you get another, and if multiply any vector in that line by a scalar, then the result is also in that line. Thus, every line through the origin is a subspace of the plane.

Furthermore, there aren't any other subspaces of the plane. At this point in our investigations, we haven't got any theorems about subspaces, so it's fairly complicated to show there aren't any more. Once we have theorems about dimensions of vector spaces, it will be easy, so we won't do that now.

Subspaces of space $\mathbf{R}^3$. We can identify the subspaces of $\mathbf{R}^3$ like we did for $\mathbf{R}^2$, but, again, we won't show they're all the subspaces since it will be easier to do that after we have more theorems.

There is, of course, the trivial subspace $\mathbf{0}$ consisting of the origin $\mathbf{0}$ alone. And $\mathbf{R}^3$ is a subspace of itself. Next, to identify the proper, nontrivial subspaces of $\mathbf{R}^3$.

Every line through the origin is a subspace of $\mathbf{R}^3$ for the same reason that lines through the origin were subspaces of $\mathbf{R}^2$.

The other subspaces of $\mathbf{R}^3$ are the planes passing through the origin. Let W be a plane passing through $\mathbf{0}$. We need (1) $\mathbf{0} \in W$, but we have that since we're only considering planes that contain $\mathbf{0}$. Next, we need (2) W is closed under vector addition. If $\mathbf{w}_1$ and $\mathbf{w}_2$ both belong to W, then so does $\mathbf{w}_1 + \mathbf{w}_2$ because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane W. Finally, we need (3) W is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in W. Thus, each plane W passing through the origin is a subspace of $\mathbf{R}^3$.

Solutions of systems of homogeneous linear equations. A polynomial is said to be *homogeneous* if all its terms have the same degree. For example $4x^3 + 5x^2y - 8xyz$ is a homogeneous cubic polynomial, whereas $4x^3 + 5xy$ is a cubic polynomial which is not homogeneous because it has a quadratic term. The word homogeneous comes from the Greek and means of the same kind.

Likewise, an equation is *homogeneous* if its terms have the same degree. $4x^3 = 5y^2z$ is homogeneous, but $4x^3 = 5xy$ is not. A linear equation is one whose terms are all degree 1 or less, and it's homogeneous if all its terms are degree 1. Here's a typical system of homogeneous equations

$$\left\{ \begin{array}{rcrcrcl} 4x & + & 2y & - & z & = & 0 \\ x & - & y & - & 4z & = & 0 \\ 2x & & & + & 3z & = & 0 \end{array} \right.$$

The set of solutions V to any system of homogeneous linear equations is a vector space. That's because $\mathbf{0} = (0, 0, 0)$ is always a solution, if $\mathbf{v}$ is a solution, then any scalar multiple of $\mathbf{v}$ is a solution, and if both $\mathbf{v}$ and $\mathbf{w}$ are solutions, then so is $\mathbf{v} + \mathbf{w}$.

Intersections of subspaces are subspaces. We'll prove that in a moment, but first, for an example to illustrate it, take two distinct planes in $\mathbf{R}^3$ passing through $\mathbf{0}$. Their intersection is a line passing through $\mathbf{0}$, so it's a subspace, too.

Theorem 3. The intersection of two subspaces of a vector space is a subspace itself.

We'll develop a proof of this theorem in class.

Note that the union of two subspaces won't be a subspace (except in the special case when one happens to be contained in the other, in which case the union is the larger one). For an example of that, consider the x-axis and the y-axis in $\mathbf{R}^2$. They're both subspaces of $\mathbf{R}^2$, but their union is not a subspace of $\mathbf{R}^2$. Why not?

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